

ON LI-YORKE MEASURABLE SENSITIVITY

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ABSTRACT. The notion of Li-Yorke sensitivity has been studied extensively in the case of topological dynamical systems. We introduce a measurable version of Li-Yorke sensitivity, for nonsingular (and measure-preserving) dynamical systems, and compare it with various mixing notions. It is known that in the case of nonsingular dynamical systems, ergodic Cartesian square implies double ergodicity, which in turn implies weak mixing, but the converses do not hold in general, though they are all equivalent in the finite measure-preserving case. We show that for nonsingular systems, ergodic Cartesian square implies Li-Yorke measurable sensitivity, which in turn implies weak mixing. As a consequence we obtain that, in the finite measure-preserving case, Li-Yorke measurable sensitivity is equivalent to weak mixing. We also show that with respect to totally bounded metrics, double ergodicity implies Li-Yorke measurable sensitivity, and extend the known result that weak mixing implies measurable sensitivity for finite measure-preserving systems to the case of infinite measure-preserving systems.

1. INTRODUCTION

The notion of sensitive dependence for topological dynamical systems has been studied by many authors, see, for example, the works [5, 9, 3] and the references therein. Recently, various notions of measurable sensitivity have been explored in ergodic theory, see for example [2, 11, 7, 13, 12, 10].

In this paper we are interested in formulating a measurable version of the topological notion of Li-Yorke sensitivity for the case of nonsingular and measure-preserving dynamical systems.

In Section 2, we review some preliminary definitions and introduce the notion of Li-Yorke measurable sensitivity (also called Li-Yorke M-sensitivity) which is based on the topological notion of Li-Yorke sensitivity in [4] and prove that in the conservative ergodic case it implies W-sensitivity introduced in [10]. In Section 3 we prove that if the Cartesian square is conservative ergodic (a condition stronger than weak mixing in the nonsingular case [1]) then it is Li-Yorke M-sensitive. Section 4 shows that for conservative ergodic nonsingular systems, ergodic Cartesian square implies Li-Yorke M-sensitivity, which in turn implies weak mixing; as consequence of this is that in the finite measure-preserving case Li-Yorke sensitivity is equivalent to weak mixing. Section 5 studies scrambled sets. In Section 6 we prove that for conservative ergodic infinite measure-preserving transformations weak mixing implies W-measurable sensitivity; this remains open in

the general nonsingular case. The final sections study entropy and the existence of a W -measurably sensitive compatible metric for any conservative ergodic transformation.

1.1. Acknowledgements. This paper is based on research by the Ergodic Theory group of the 2011 SMALL summer research project at Williams College. Support for the project was provided by National Science Foundation REU Grant DMS - 0353634 and the Bronfman Science Center of Williams College. We would like to thank Sergiy Kolyada for bringing [4] to our attention.

2. PRELIMINARY DEFINITIONS AND MEASURABLE SENSITIVITY

A *nonsingular dynamical system* (X, \mathcal{S}, μ, T) is a standard Borel space (X, \mathcal{S}) with a σ -finite, nonatomic measure μ and nonsingular endomorphism $T : X \rightarrow X$ (i.e., for all $A \in \mathcal{S}$, $T^{-1}(A) \in \mathcal{S}$ and $\mu A = 0$ if and only if $\mu(T^{-1}A) = 0$). We sometimes take T to be measure-preserving or the measure space to be finite. We sometimes suppress \mathcal{S} and write (X, μ, T) . We say that T is conservative if for all A of positive measure there exists $n > 0$ such that $\mu(T^{-n}A \cap A) > 0$, and it is ergodic if whenever $T^{-1}A = A$ then $\mu(A) = 0$ or $\mu(X \setminus A) = 0$. A nonsingular transformation T is weakly mixing if whenever f is an L^∞ function such that $f \circ T = zf$ for $z \in \mathbb{C}$, then f is constant a.e. If $T \times T$ is ergodic then T is weakly mixing, but the converse, while true in the finite measure-preserving case, does not hold in general [1].

We consider metrics d on X . We assume throughout that these are Borel measurable and bounded by 1. We say a metric d on X is μ -compatible if μ assigns positive measure to nonempty, open d -balls [13, 10]. It follows from Proposition 2.1 in [10] that the topology generated by d is separable. It follows that open sets are measurable as they are countable unions of balls. The notion of measurable sensitivity was introduced in [13].

Definition 2.1. We say a nonsingular dynamical system (X, μ, T) is measurably sensitive if for every isomorphic mod 0 dynamical system (X_1, μ_1, T_1) and μ_1 -compatible metric d on X_1 there exists a $\delta > 0$ such that for all $x \in X_1$ and $\varepsilon > 0$ there is an $n \in \mathbb{N}$ such that

$$\mu_1\{y \in B_\varepsilon(x) : d(T_1^n(x), T_1^n(y)) > \delta\} > 0.$$

This definition was refined in [10].

Definition 2.2. Let (X, μ, T) be a nonsingular dynamical system and d a μ -compatible metric on X . We say the system is W -measurably sensitive with respect to d if there is a $\delta > 0$ such that for each $x \in X$

$$\limsup_{n \rightarrow \infty} d(T^n x, T^n y) > \delta$$

for a.e. $y \in X$. The system is W -measurably sensitive if it is W -measurably sensitive with respect to each μ -compatible metric d .

Proposition 7.2 in [10] shows that these two definitions are equivalent in the conservative ergodic case. Requiring that the condition hold for each $x \in X$ and a.e. $y \in X$ is equivalent to requiring that it hold for a.e. pair $(x, y) \in X^2$ (Proposition A.1, [10]), a notion called pairwise sensitivity introduced in [7]. In [10] they have the following classification result (Theorem 7.1).

Proposition 2.1. *Let (X, μ, T) be a conservative ergodic nonsingular dynamical system. Then T is W -measurably sensitive or T is isomorphic mod 0 to an invertible minimal uniformly rigid isometry on a Polish space.*

The proof of this theorem shows that the isometric metric is compatible. The following are two technical propositions from [10] that are needed for our work. The proofs follow those in the aforementioned paper and are included for completeness.

Proposition 2.2. *Suppose T is a nonsingular transformation. If for almost every pair $(x, y) \in X \times X$ there exists $n \geq 0$ such that $d(T^n x, T^n y) \geq \delta$, then for almost every pair $(x, y) \in X \times X$ we have $\limsup_{n \rightarrow \infty} d(T^n x, T^n y) \geq \delta$.*

Proof. Let

$$Z(N, x) = \{y \in X : \exists n, d(T^n(T^N x), T^n y) \geq \delta\}.$$

Then by hypothesis there exists a full measure set A such that $\mu(Z(N, x)^c) = 0$ for each $x \in A$ and for all $N \in \mathbb{N}$. Let

$$Y(N, x) = \{y \in X : \exists n > N, d(T^n x, T^n y) \geq \delta\}.$$

Note that $Y(N, x) = T^{-N}(Z(N, x))$. Since $Z(N, x)$ has full measure and T is nonsingular we see that $Y(N, x)$ also has full measure. Hence

$$\bigcap_{N > 0} Y(N, x)$$

also has full measure. But this says for each $x \in A$ and almost every $y \in X$ that $\limsup_{n \rightarrow \infty} d(T^n x, T^n y) \geq \delta$. \square

The proof of the next proposition follows from the same arguments as above.

Proposition 2.3. *Suppose T is a nonsingular transformation. If for almost every pair $(x, y) \in X \times X$ there exists $n \geq 0$ such that $d(T^n x, T^n y) \leq \delta$, then for almost every pair $(x, y) \in X \times X$ we have $\liminf_{n \rightarrow \infty} d(T^n x, T^n y) \leq \delta$.*

The notion of W -measurable sensitivity adapts the notion of sensitivity to initial conditions from topological dynamics to the measurable case. In topological dynamics there is also the notion of Li-Yorke sensitivity, in which points are not only required to separate but also to come back together. We give the definitions of topological Li-Yorke sensitivity as in [4].

Definition 2.3. *Let (X, d, T) be a topological dynamical system. A pair (x, y) is said to be proximal if*

$$\liminf_{n \rightarrow \infty} d(T^n x, T^n y) = 0.$$

Definition 2.4. Let (X, d, T) be a topological dynamical system. We call the system Li-Yorke sensitive if there exists an $\varepsilon > 0$ such that every $x \in X$ is a limit of points $y \in X$ such that the pair (x, y) is proximal but whose orbits are at least ε apart at arbitrarily large times.

In this paper we consider the measure-theoretic analogue of Li-Yorke sensitivity.

Definition 2.5. Let (X, μ, T) be a nonsingular dynamical system and d a μ -compatible metric on X . We say that a pair (x, y) is a Li-Yorke pair if

$$\liminf_{n \rightarrow \infty} d(T^n x, T^n y) = 0 \text{ and } \limsup_{n \rightarrow \infty} d(T^n x, T^n y) > 0.$$

We say (X, μ, T) is Li-Yorke measurably sensitive for the metric d if the set of Li-Yorke pairs $(x, y) \in X \times X$ has full measure. We say it is Li-Yorke measurably sensitive (henceforth Li-Yorke M-sensitive) if it is Li-Yorke M-sensitive for all μ -compatible metrics.

Note that atomic measures are never sensitive. We now show that Li-Yorke M-sensitivity is a (measurable) isomorphism invariant. We need the following lemma ([10], Lemma 6.1).

Lemma 2.4. Let (X, \mathcal{S}) be a standard Borel space with nonatomic measure μ . Let $U \subset X$ be a Borel subset of full measure and d_U a μ -compatible metric defined on U . Then d_U can be extended to a μ -compatible metric d_X on X such that d_U and d_X agree on a set of full measure subset contained in $U \times U$.

The fact that the set is contained in $U \times U$ is not in the statement but follows from the proof of the lemma in the original paper. We now proceed in a way similar to Proposition 6.2 of the same paper.

Proposition 2.5. Suppose nonsingular dynamical system (X, μ, T) is Li-Yorke M-sensitive. Then any isomorphic system (Y, ν, S) is also Li-Yorke M-sensitive.

Proof. Suppose (Y, ν, S) is not Li-Yorke M-sensitive. Then there is a ν -compatible metric d_Y on Y for which (Y, ν, S) is not Li-Yorke M-sensitive. As the systems are isomorphic, there are Borel sets $U \subset X$ and $V \subset Y$ of full measure and a bijection $\phi : U \rightarrow V$ such that $\phi \circ T = S \circ \phi$. Define a μ -compatible metric d_U on U by $d_U(x, y) = d_Y(\phi(x), \phi(y))$. Applying Lemma 2.4 extends d_U to a μ -compatible metric d_X on X which agrees with d_U on a set $X_0 \subset U \times U$ of full measure in $X \times X$. By hypothesis T is Li-Yorke M-sensitive, so the set $L \subset X^2$ of Li-Yorke pairs has full measure. It follows that the set $A = \bigcap (T \times T)^{-n}(X_0 \cap L)$ has full measure. Note that if $(x, y) \in A$, it is Li-Yorke, and for all n we have that $d_U(T^n x, T^n y) = d_X(T^n x, T^n y)$ and $\phi(T^n x), \phi(T^n y) \in V$ (since $T^n x, T^n y \in U$). Now $\phi \times \phi(A)$ has full measure in $Y \times Y$, and for all $(\phi(x), \phi(y)) \in \phi \times \phi(A)$ we have for all n that

$$d_Y(S^n \phi(x), S^n \phi(y)) = d_U(T^n x, T^n y) = d_X(T^n x, T^n y).$$

It follows that all pairs in $\phi \times \phi(A)$ are Li-Yorke for d_Y , a contradiction. \square

Li-Yorke M-sensitivity is not a priori stronger than W-measurable sensitivity. But this turns out to be the case when the system in question is conservative ergodic.

Proposition 2.6. *Let (X, μ, T) be a conservative ergodic and nonsingular dynamical system. If it is Li-Yorke M-sensitive, then it is W-measurably sensitive.*

Proof. We show the contrapositive. If T is not W-measurably sensitive, then by Theorem 2.1 it is isomorphic mod 0 to an isometry. But then the isomorphic system is both Li-Yorke M-sensitive and an isometry for a compatible metric, which is impossible. \square

3. MIXING CONDITIONS STRONGER THAN LI-YORKE SENSITIVITY

We study not just the \liminf and \limsup but also the set of distances to which T separates pairs. Define

$$\mathcal{N} = \{r : \text{for a.e. } (x, y) \in X^2, \exists \{n_k\} \text{ s.t. } d(T^{n_k}x, T^{n_k}y) \rightarrow r\}.$$

Notice that $0 \in \mathcal{N}$ for Li-Yorke M-sensitive systems. Further, it is easily shown that \mathcal{N} is closed. We then have the following result. Let G be the set of values taken on by the metric d .

Proposition 3.1. *Let $T \times T$ be conservative ergodic nonsingular and d be a compatible metric on (X, μ) . Then for a.e. $(x, y) \in X^2$, we have that $(T^n x, T^n y)$ is dense in the product topology, so that for any $r \in G$ there is an $\{n_k\}$ such that $d(T^{n_k}x, T^{n_k}y) \rightarrow r$.*

Proof. Since (X, d) is separable it has a countable topological basis $\{B_i\}$ of balls. As the metric is μ -compatible, $B_i \times B_j$ has positive measure in the product space for each i and j . It then follows from the ergodicity of $T \times T$ that

$$A = \bigcap_{i,j} \bigcup_n (T \times T)^{-n}(B_i \times B_j)$$

has full measure. By construction $(T^n x, T^n y)$ is dense in the product topology for all $(x, y) \in A$. Now for each k one can choose n_k such that $T^{n_k}x \in B(z_1, 1/k)$ and $T^{n_k}y \in B(z_2, 1/k)$ where $d(z_1, z_2) = r$. \square

Corollary 3.2. *Under the hypotheses of Proposition (3.1) $\overline{G} \subset \mathcal{N}$. In particular, T is Li-Yorke M-sensitive and W-measurably sensitive.*

We now investigate another condition under which T is Li-Yorke M-sensitive. The ergodicity of $T \times T$ is a strong condition in general. We can extend the results from the $T \times T$ ergodic case to similar results in the doubly ergodic case. The main difference is the necessity of requiring the metric to have some degree of compactness. Define H to be the set of $r \in G$ such that for every ball B there is a point $x \in B$ and $y \in X$ such that $d(x, y) = r$. Note that $0 \in H$.

Proposition 3.3. *Let T be doubly ergodic and let d be a compatible metric such that (X, d) is compact. For each $r \in H$, for all $x \in X$, for a.e. $y \in X$ there is a sequence $\{n_k\}$ such that $d(T^{n_k}x, T^{n_k}y) \rightarrow r$.*

Proof. Suppose to the contrary. Then there is some $r \in H$, $x \in X$ and positive measure set $B \subset X$ such that for no y in B is it the case that $d(T^{n_k}x, T^{n_k}y) \rightarrow r$ for any sequence $\{n_k\}$. Then for each $y \in B$ there is an N and an ε such that $|d(T^n x, T^n y) - r| \geq \varepsilon$ for all $n \geq N$. Thus we can find an N and an ε for which this property holds on a positive measure subset of B (which for simplicity we just relabel B). Cover X with a finite number of $\varepsilon/2$ -balls B_1, \dots, B_k . As T is doubly ergodic, it is k -ergodic ([6]), so we can choose an n so that $\mu(T^{-n}B_i \cap B) > 0$ for each $i = 1, \dots, k$. We have that $T^n x$ is contained in some ball B_j , and by hypothesis this ball contains a point x' such that for some y' in the space $d(x', y') = r$. Now y' is contained in some ball B_ℓ . Consider any $y \in T^{-n}B_\ell \cap B$. Then $r - \varepsilon < d(T^n x, T^n y) < r + \varepsilon$ for all $y \in B'$, a contradiction. \square

Lemma 3.4. *If T is doubly ergodic then it does not admit a compatible metric for which it is an isometry.*

Proof. Suppose not and let d be a compatible isometric metric. Choose an $\varepsilon > 0$ such that there exists sets $A, B \subset X$ with $d(A, B) > \varepsilon$. Since d is compatible $\mu(B_{\varepsilon/2}(x)) > 0$. By double ergodicity there exists an n such that

$$\mu(T^{-n}(B_{\varepsilon/2}(x)) \cap A) > 0 \quad \mu(T^{-n}(B_{\varepsilon/2}(x)) \cap B) > 0.$$

This implies that the diameter of $T^n(B_{\varepsilon/2}(x))$ is strictly greater than ε contradicting the fact that T is an isometry for d . \square

Corollary 3.5. *Let T be doubly ergodic and d be a totally bounded compatible metric. Then T is Li-Yorke M -sensitive with respect to d .*

Proof. By Proposition 6.3 [10] we know that T is either W -measurably sensitive or is measurably isomorphic to an isometry. Since it is doubly ergodic the preceding lemma shows that it cannot be isomorphic to an isometry and hence is W -measurably sensitive. Thus T separates points.

We now show that T brings points together. Suppose it did not. Proceeding as in the proof of Proposition 2.1 (Theorem 7.1 in [10]), let (X_1, d_1) be the topological completion of (X, d) . We can extend the measure to this space by letting sets $S \subset X_1$ be measurable when $S \cap X$ is measurable and by defining $\mu_1(S) = \mu(S \cap X)$. Define T_1 on X_1 as T on X and as the identity on $X_1 \setminus X$. This new system is isomorphic mod 0 to (X, μ, T) , so T is doubly ergodic. Since (X, d) is totally bounded, (X_1, d_1) is compact. Then by Proposition 3.3, T_1 brings points together in the metric d_1 . It follows that T brings points together in the metric d . \square

4. LI-YORKE M -SENSITIVITY AND WEAK MIXING

We now show that Li-Yorke sensitivity implies weak mixing, which will be a converse to the above results in the finite measure-preserving case. We

first show the useful fact that every positive measure subset of a standard Borel space admits a compatible metric, which in fact turns out to be totally bounded. We start with a lemma whose proof is standard.

Lemma 4.1. *Given a measure space (X, \mathcal{S}, μ) with a σ -finite measure μ there exists an equivalent probability measure η .*

Proposition 4.2. *Let (X, \mathcal{B}, μ) be a standard Borel space with non-atomic σ -finite measure μ . Then there exists a totally bounded compatible metric d on X .*

Proof. By Lemma 4.1 there exists an equivalent probability measure η . Now using the isomorphism theorem for measures ([14], Theorem 17.41), there is a Borel isomorphism $f : X \rightarrow [0, 1]$ with $f\eta = m$ where m is the Lebesgue measure on $[0, 1]$. Define a metric d on X by

$$d(x, y) = |f(x) - f(y)|.$$

Note that this defines a totally bounded metric. Now we show it is compatible. Given $x \in X$ and $\varepsilon > 0$ consider $B_\varepsilon(x)$. The set $f(B_\varepsilon(x))$ is an open ball of radius ε about $f(x)$ on the interval $[0, 1]$ and thus has positive Lebesgue measure. As $m(f(B_\varepsilon(x))) > 0$, we have that $\eta(B_\varepsilon(x)) > 0$, so that $\mu(B_\varepsilon(x)) > 0$. Hence d is compatible. \square

Corollary 4.3. *Let (X, \mathcal{B}, μ) be a standard Borel space with non-atomic σ -finite measure μ and $Y \subset X$ be a positive measure subset of X . Then there exists a totally bounded compatible metric d on Y .*

Proof. By Corollary 13.4 ([14]) we know that Y is still a standard Borel space. Note that the restriction of μ to Y is still a non-atomic non-trivial σ -finite measure. Hence the result follows by Proposition 4.2. \square

We now show the main result of this section.

Proposition 4.4. *Let (X, μ, T) be a conservative ergodic nonsingular dynamical system. If T is Li-Yorke M -sensitive, then it is weakly mixing.*

Proof. We prove the contrapositive. Suppose T is not weakly mixing, so that it has a nonconstant eigenfunction $f : X \rightarrow S^1$. This is a factor map mapping (X, μ, T) to (S^1, ν, R) where ν is some measure on the circle and R is a rotation. Given $z, z' \in S^1$ let $a(z, z')$ denote the normalized arc length on the circle (so $a(1, -1) = \frac{1}{2}$). Since the map f is non-trivial and measure-preserving there exists an m such that the set $A \subset X^2$ defined by

$$A = \{(x, y) \in X \times X : a(f(x), f(y)) \geq 1/m\}$$

has positive measure. Let $D_i = f^{-1}([e^{\frac{2\pi(i-1)}{m}}, e^{\frac{2\pi i}{m}}))$ for $i = 1, \dots, m$. Relabel the D_i as

$$\{E_1, \dots, E_k, \dots, E_m\},$$

where $\mu(E_j) = 0$ for $j > k$. Then we have

$$X = (E_1 \sqcup \dots \sqcup E_k) \mod 0.$$

By removing from X the backwards orbits of E_j for $j > k$, a measure zero set, we obtain a T -invariant set of full measure. Thus since Li-Yorke M-sensitivity is preserved under measurable isomorphism we can assume that $X = E_1 \sqcup \dots \sqcup E_k$. By Corollary 4.3 we know that for each $j \leq k$ there exists a metric d_j on E_j which is compatible. Define a metric d on X by

$$d(x, y) = \begin{cases} 1 & \text{if } x \in E_i, y \in E_j, i \neq j \\ d_i(x, y) & \text{if } x, y \in E_i. \end{cases}$$

Note that d is a compatible metric for X . As A has positive measure in the product space, showing that no $(x, y) \in A$ is proximal suffices to show that T is not Li-Yorke M-sensitive. Note that since R is a rotation it is an isometry under arc length. Then if $(x, y) \in A$,

$$a(f(T^n x), f(T^n y)) = a(R^n f(x), R^n f(y)) = a(f(x), f(y)) \geq 1/m.$$

This implies that $T^n x$ and $T^n y$ cannot lie in the same E_i , for otherwise we would have that $a(f(x), f(y)) < 1/m$. Thus $d(T^n x, T^n y) = 1$ for all $n \geq 0$, which implies (x, y) is not proximal. \square

Collecting our results we see that we have the following chain of implications.

Theorem 1. *Let (X, μ, T) be a conservative ergodic nonsingular dynamical system. Then*

$$T \times T \text{ ergodic} \Rightarrow T \text{ Li-Yorke M-sensitive} \Rightarrow T \text{ weakly mixing}.$$

In the finite measure-preserving case it is known that weak mixing and ergodicity of the product are equivalent. We thus have the following result.

Theorem 2. *Suppose (X, T, μ) is a finite measure-preserving ergodic dynamical system. Then T is Li-Yorke M-sensitive if and only if it is weakly mixing.*

Note that in this case T is not just Li-Yorke M-sensitive but in fact separates points arbitrarily close to every possible distance that any given compatible metric assumes. We now show the following lemma, which will allow us to relate this result to the classification result on W-measurable sensitivity (Proposition 2.1).

Lemma 4.5. *Suppose (X, μ, T) is an ergodic finite measure-preserving transformation. Then T is weakly mixing if and only if T has no non-trivial factors which admit a compatible metric for which they are an isometry.*

Proof. If T is weak mixing, then all its factors are doubly ergodic and so by Lemma 3.4 do not admit compatible isometric metrics. If T is not weak mixing, then it has a non-trivial rotation factor. If the rotation is irrational, then the measure in the factor must be Lebesgue measure, for which arc length is an isometry. Otherwise, the factor is a rotation on n points. Spacing these atoms evenly around the unit circle and using arc length gives an isometric metric. \square

Corollary 4.6. *Suppose (X, μ, T) is a finite measure-preserving ergodic dynamical system. Then T is Li-Yorke M -sensitive if and only if no non-trivial factors of T admit a compatible isometric metric.*

This makes the connection to [10] and especially to Proposition 2.1 very clear.

5. SCRAMBLED SETS

In the topological case, Akin and Kolyada [4] define the notion of a scrambled set. They take the existence of an uncountable scrambled set as indicating that a system is chaotic. Here we show that Li-Yorke M -sensitive systems always contain an uncountable scrambled set.

Definition 5.1. *A set $A \subset X$ is said to be scrambled if for all $x, y \in A$ with $x \neq y$ the pair (x, y) is Li-Yorke.*

Proposition 5.1. *Let $A \subset X^2$ have positive measure and suppose there exists $D \subset X$ with positive measure such that $A \subset D \times D$ and A has full measure in $D \times D$. Then for almost every $x \in D$ there exists an uncountable set $B \subset X$ containing x such that $y, z \in B$ with $y \neq z$ implies $(y, z) \in A$.*

Proof. In this context we use the term scrambled set to refer to a set C such that $y, z \in C$ and $y \neq z$ implies $(y, z) \in A$. We can assume without loss of generality that $D^2 = X^2$ and that A has full measure. Then a.e. $x \in X$ has a full-measure fiber, i.e. $\mu\{y \in X : (x, y) \in A\} = 1$. Let the set of such x be denoted X_0 . For any $x, y \in X_0$ let P be the collection of scrambled sets contained in X_0 and containing $\{x, y\}$. This is nonempty as $\{x, y\} \in P$. Let P be partially ordered by set inclusion and take Q to be any totally ordered subset. Then $\bigcup_{S \in Q} S$ is contained in P and is an upper bound for Q . Thus every totally ordered subset of P has an upper bound. Now Zorn's Lemma guarantees the existence of an element M such that $M \in P$ and M is not contained in any other element of P . We show M is uncountable. Suppose it were not. Then for $z \in A$ let $C(z) = \{w \in X : (z, w) \text{ is Li-Yorke}\}$. We have $\mu(C(z)^c) = 0$ so that $\mu((\bigcap_{z \in A} C(z))^c) = 0$. In particular $\bigcap_{z \in A} C(z)$ is nonempty and so contains some w . But then $M' = M \cup \{w\}$ is scrambled and $M \subsetneq M'$. \square

Corollary 5.2. *Suppose T is a Li-Yorke M -sensitive nonsingular transformation. Then a.e. x is contained in an uncountable scrambled set.*

Thus a Li-Yorke M -sensitive transformation is also chaotic in the sense of [4] when viewed as a topological dynamical system under any compatible metric. Uncountable scrambled sets play a smaller role here than they do in topological dynamics because uncountable sets can still be measure-theoretically small. However, this idea will be important in proving that periodic transformations admit no compatible sensitive metrics.

6. WEAK MIXING AND W-MEASURABLE SENSITIVITY

Weak mixing implies W-measurable sensitivity in the finite measure-preserving case, as in this case weak mixing is equivalent to double ergodicity. We now show that this result still holds in the infinite measure-preserving case. We first have the following lemma. The result is similar to one in the proof of Theorem 2 in [10], although the idea of the proof is a bit different.

Lemma 6.1. *Let (X, μ, T) be a conservative ergodic measure-preserving dynamical system and d a compatible metric on X for which T is an invertible isometry. Then the measure of balls $B_r(x)$ depends only on the radius.*

Proof. By Lemma 5.4 in [10] we know that almost every point of X is transitive. Let x be a transitive point. Fix $y \in X$. Then for all $\varepsilon > 0$ there exists an n such that $d(T^n x, y) < \varepsilon$. Then $T^n(B_{r-\varepsilon}(x)) \subset B_r(y)$ so that

$$\mu(B_{r-\varepsilon}(x)) = \mu(T^n(B_{r-\varepsilon}(x))) \leq \mu(B_r(y)).$$

Similarly $B_r(y) \subset T^n(B_{r+\varepsilon}(x))$ so that

$$\mu(B_{r+\varepsilon}(x)) = \mu(T^n(B_{r+\varepsilon}(x))) \geq \mu(B_r(y)).$$

Since $\varepsilon > 0$ was arbitrary and the measure is nonatomic, these two inequalities together show that $\mu(B_r(y)) = \mu(B_r(x))$. \square

Theorem 3. *Let (X, μ, T) be a conservative ergodic measure-preserving dynamical system. If T is weakly mixing, T is W-measurably sensitive.*

Proof. We argue the contrapositive. If T is not W-measurably sensitive then by Proposition 2.1 it is isomorphic mod 0 to an invertible isometry on a Polish space whose metric is compatible. Relabel this system (X, μ, T) with metric d_X . Proposition 5.6 in [10] and the discussion preceding it demonstrate the following. Let G be the set of continuous transformations on X which commute with T . Then G is a group. For $S_1, S_2 \in G$ define $d(S_1, S_2) = \sup_{x \in X} d_X(S_1 x, S_2 x)$. Then d is a metric on G and G is equal to the closure of the set $\{\mathbb{I}, T, T^2, \dots\}$. Moreover, for any $x \in X$ the map $\phi_x : G \rightarrow X$ given by $\phi_x(S) = Sx$ is an isometry. Choose one such map ϕ and use it to induce a measure η on G . Let $T : G \rightarrow G$ denote the group rotation on G by the element T . Then (G, η, T) is isomorphic mod 0 to (X, μ, T) . Let \mathcal{B} be the induced σ -algebra on G . We now establish some properties of G .

Notice first that since ϕ_x is an isometry for all x , $d(S_1 x, S_2 x) = d(S_1, S_2) = d(S_1 y, S_2 y)$ for all $x, y \in X$. It then follows that a right cancellation law holds:

$$d(S_1 S, S_2 S) = d(S_1 Sx, S_2 Sx) = d(S_1(Sx), S_2(Sx)) = d(S_1, S_2)$$

for $S, S_1, S_2 \in G$. Now T is an isometry, and since G is the closure of the orbit of T , for every map $S \in G$ there is a sequence T^{n_k} which converges uniformly to S . It follows that S is an isometry, i.e.,

$$d(SS_1, SS_2) = d(S_1, S_2)$$

for $S, S_1, S_2 \in G$.

We now show that G is Abelian. Let $S_1, S_2 \in G$ and $\varepsilon > 0$. Choose k so that $d(T^k S_1, S_2) < \varepsilon/2$. Then

$$d(S_1 S_2, S_2 S_1) \leq d(T^k S_1 S_2, S_2 S_2) + d(S_2 S_2, S_2 T^k S_1) = 2d(T^k S_1, S_2) < \varepsilon.$$

It follows that $S_1 S_2 = S_2 S_1$.

We now show that every $S \in G$ is a nonsingular transformation on X with respect to μ . As S is an isometry, it maps a ball of radius r to another ball of radius r . We showed in the preceding lemma that all such balls have the same measure. Thus for all balls B we have that $\mu_S(B) = \mu(S^{-1}(B)) = \mu(B)$. Since d is μ -compatible, X is separable under d . Hence there exists a countable dense subset $\{x_1, x_2, \dots\}$. Now consider the collection of balls of rational radius about each x_i . Call this family \mathcal{C} . Then this is a countable family of Borel subsets which separate the points of X . Hence by the Blackwell–Mackey Theorem [15], \mathcal{C} is a generating family for \mathcal{B} . Since μ and μ_S agree on a generating family they agree on all of \mathcal{B} . Thus each $S \in G$ is nonsingular (and in fact measure-preserving) for μ .

We now show that G is locally compact. Note that the fact that each $S \in G$ is nonsingular for μ implies that multiplication by S in the group is nonsingular. Let \mathcal{C}_η be the measure class of η in G . Then we have just seen that it is invariant under G . In addition since ϕ is an isometry the fact that all balls of a given radius in X have the same measure implies that all balls in G of a given radius have the same measure. We now show G is locally compact. Given an element $S \in G$ choose $r > 0$ so that $E = \bar{B}_r(S)$, the closed ball of radius r about S , has finite measure. Let $m = \eta(E)$. Since E is a closed subset of a complete space it is complete. Now choose $\varepsilon > 0$. Then let $\alpha = \eta(B_\varepsilon(S))$. Note that we can fit at most $\frac{m}{\alpha}$ disjoint ε balls into E . Start filling E with balls of radius ε until you cannot fit another ε ball disjoint from all the rest into E . Note that we have just shown that this process will terminate. Let $B_\varepsilon(S_1), \dots, B_\varepsilon(S_n)$ be the ε balls. Note that given any $S' \in E$ it must be that there exists an i such that $d_G(S', S_i) < 2\varepsilon$. Then

$$E \subset B_{2\varepsilon}(S_1) \cup \dots \cup B_{2\varepsilon}(S_n)$$

so that E is totally bounded. Hence E is totally bounded and complete and hence compact. Thus the group G is locally compact.

Now Theorem 8.3.4 in [8] tells us that G is a Maximally Almost Periodic group (MAP) which implies that G has sufficiently many characters. Hence there exists a character ξ which separates T and the identity. Thus $\xi(T) = \lambda \neq 1$. We show that this character is non-trivial with respect to the measure. Suppose it were so that there existed a point $z \in S^1$ such that $\xi^{-1}(z)$ has full measure. Then since T is nonsingular we know that $T\xi^{-1}(z)$ also has full measure. If $S' \in T\xi^{-1}(z)$ then we can write $S' = TS$ for $S \in \xi^{-1}(z)$. But then since $\lambda \neq 1$ we see that

$$\xi(S') = \xi(TS) = \xi(T)\xi(S) = \lambda z \neq z$$

so that $\xi^{-1}(z)$ and $T\xi^{-1}(z)$ are disjoint full measure sets, a contradiction. Thus ξ is nontrivial with respect to the measure. Multiplication by T in G corresponds to rotation by λ under ξ , so ξ is a nontrivial factor map. Thus the group rotation by T in G has a non-trivial rotation factor and hence cannot be weakly mixing. Since (G, T, η) and (X, T, μ) are measurably isomorphic, (X, T, μ) is also not weakly mixing. \square

This proof differs significantly from that in the finite measure-preserving case. This is because in that case one exploits a highly nontrivial equivalence between weak mixing and the ergodicity of the product. Showing that the group is locally compact is a fundamental part of the proof. This is lost in the nonsingular case.

7. ENTROPY AND SENSITIVITY

Sensitivity is only one way to capture the notion of chaos. It is generally accepted that entropy should be stronger than sensitivity notions. We show that positive entropy implies WM-sensitivity, improving a result of Cadre and Jacob [7]. The preceding sections suggest that Li-Yorke M-sensitivity is a fundamentally stronger concept than W-measurable sensitivity. Here we demonstrate examples of transformations which are W-measurable sensitive but not Li-Yorke M-sensitive.

Proposition 7.1. *Let T on (X, μ) be an ergodic finite measure-preserving transformation. If T has positive entropy, it is W-measurably sensitive.*

Proof. We show the contrapositive. If T is not W-measurably sensitive then we know it is measurably isomorphic to a compact group rotation ([10], Theorem 2). By the exact same argument as in the proof of Theorem 3 this group is Abelian. It is well known that compact abelian group rotations have zero entropy ([16], Theorem 4.25). \square

Corollary 7.2. *All finite measure-preserving ergodic transformations which have positive entropy but which are not weak mixing are W-measurably sensitive and not Li-Yorke M-sensitive.*

Remark. We now construct examples of W-measurably sensitive but not Li-Yorke M-sensitive transformations with zero entropy. Let (X, μ, T) be a finite measure-preserving doubly ergodic transformation with zero entropy and let (Y, ν, R) be the rotation on two points. Consider the dynamical system $(X \times Y, \mu \times \nu, T \times R)$. This clearly has zero entropy. We show this system is W-measurably sensitive and not Li-Yorke M-sensitive. Let $S = T \times R$. To see it is not Li-Yorke M-sensitive, one could note that it has a rotation factor and apply Theorem 2. More explicitly, one could consider any μ -compatible metric d_X on X and the compatible metric d_Y on Y which has $d_Y(1, 2) = 1$. Then define d on $X \times Y$ by

$$d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2).$$

It is easy to see that d is a compatible metric. If $x_1, x_2 \in X$ then

$$d(S^n(x_1, 1), S^n(x_2, 2)) \geq 1$$

for all $n \geq 0$. Hence the pair $((x_1, 1), (x_2, 2))$ is not proximal and so not Li-Yorke. Thus the set $(X \times \{1\}) \times (X \times \{2\}) \subset (X \times Y)^2$ contains no Li-Yorke pairs and has positive measure, so S is not Li-Yorke M-sensitive.

Now we show that the system is W-measurably sensitive. By Proposition 7.2 [10] it suffices to show that S is measurably sensitive. Let d be an arbitrary compatible metric on $X \times Y$. Then choose $\delta > 0$ such that $2\delta < \text{diam}(X \times \{i\})$. Choose a point $z = (x, i) \in X \times Y$ and consider $B_\varepsilon(z)$. Then we know that there must exist an i such that $\mu(B_\varepsilon(z) \cap (X \times \{i\})) > 0$. Let E be the projection of $B_\varepsilon(z) \cap (X \times \{i\})$ onto X . Note that E has positive measure. Now choose sets $A, B \subset X$ of positive measure such that $d(A, B) > 2\delta$. Since T is doubly ergodic we can find an n such that

$$\begin{aligned} \mu(T^{-n}(A) \cap E) &> 0, \\ \mu(T^{-n}(B) \cap E) &> 0. \end{aligned}$$

Then we see that $\text{diam}(T^n(E)) > 2\delta$ so that by the triangle inequality there must exist a $z' \in E$ such that $d(T^n z, T^n z') > \delta$. Hence T is measurably sensitive and thus W-measurably sensitive.

8. SENSITIVE METRICS

In the next proposition we construct a W-measurably sensitive metric for all transformations satisfying the mild hypotheses of being conservative ergodic. This motivates our previous definition requiring that sensitivity be exhibited for all compatible metrics.

Proposition 8.1. *Suppose (X, μ, T) is a conservative and ergodic transformation. Then there exists a compatible metric d for which T is W-measurably sensitive.*

Proof. If T is W-measurably sensitive then we are done, so suppose it is not. Then by Proposition 2.1 we know that there exists a compatible metric d' on X for which T is an isometry. Since d' is compatible we can partition the space X into disjoint positive measure sets $\{D_n\}$ such that $\text{diam}_{d'}(D_n) \rightarrow 0$ as $n \rightarrow \infty$. Since X is a standard Borel space we know that for each D_n there exists a compatible metric d_n . Define a compatible metric d on X by

$$d(x, y) = \begin{cases} 1, & \text{if } x \in D_i, y \in D_j, i \neq j \\ d_n(x, y), & \text{if } x, y \in D_i. \end{cases}$$

Suppose that the transformation T is not W-measurably sensitive for d . Then applying Proposition 2.2 there would exist $x \in X, \delta < 1$ and a positive measure set E such that

$$\sup_{n \geq 0} d(T^n x, T^n y) \leq \delta$$

for all $y \in E$. Now let $\alpha = \text{diam}_{d'}(E) > 0$. Then there exists n such that $\text{diam}(D_n) < \alpha$. Using the fact that T is conservative and ergodic there exists m such that $\mu(T^{-m}(D_n) \cap E) > 0$. Hence $T^m(E)$ intersects D_n . Note that since T is an isometry under d' we have that

$$\text{diam}_{d'}(E) = \text{diam}_{d'}(T^m(E)) = \alpha > \text{diam}_{d'}(D_n).$$

Hence it cannot be that $T^m(E) \subset D_n$. Thus there exists a $j \neq n$ such that $\mu(T^m(E) \cap D_j) > 0$. Suppose without loss of generality that $T^m(x) \notin D_j$. Then there exists $y \in E$ such that $T^m(y) \in D_n$. But then $d(T^m x, T^m y) = 1 > \delta$, a contradiction. Thus T must be W -measurably sensitive for d . \square

We have now shown the existence of a sensitive metric for a large class of transformations. With such a result it is important to prove that there exists a class of transformations for which it does not apply, i.e. a family of transformations which do not admit any compatible metric. Here we show that periodic transformations, by which we mean those with a positive measure invariant set X_1 such that $T^n = I$ for some n on X_1 , do not admit a W -measurably sensitive metric.

Proposition 8.2. *Let (X, μ, T) be a transformation with an invariant set X_1 such that $T^N = I$ on X_1 for some N . Then there exists no μ -compatible metric d for which T is W -measurably sensitive.*

Proof. We argue by contradiction. Suppose there exists a compatible metric d for which T is W -measurably sensitive with sensitivity constant $\delta > 0$. By restricting ourselves to X_1 we can assume that $X_1 = X$. Fix $x \in X$ and let

$$E_0 = \{y \in X : d(x, y) < \delta 2^{-N-1}\}.$$

Since the metric is compatible we know that $\mu(E_0) > 0$. We now consider two cases.

Case 1: Suppose that for each $y \in E_0$ and almost every $z \in E_0$ we have

$$d(Ty, Tz) > \delta 2^{-N-1}.$$

By an argument analogous to the one in Corollary 5.2 we can find an uncountable set $A \subset E_0$ such that $y, z \in A$ with $y \neq z$ implies $d(Ty, Tz) > \delta 2^{-N-1}$. Then the collection $\{B_{\delta 2^{-N-1}}(Ty)\}_{y \in A}$ is pairwise disjoint. But then the space contains uncountably many disjoint balls, a contradiction since it is separable under d .

Case 2: Suppose there exists a $y \in E_0$ and a positive measure set $E_1 \subset E_0$ such that for all $z \in E_1$ we have

$$d(Ty, Tz) < \delta 2^{-N}.$$

Now we are in the exact situation where we began. If ever we are in the situation of Case 1 we are done. Thus suppose we reach step $N - 2$. Then we have shown the existence of a $y \in E_0$ and a set $E_{N-2} \subset E_0$ of positive measure such that for each $z \in E_0$ we have

$$E_{N-2} = \{z : d(T^N y, T^N z) < \delta 2^{-3}\}.$$

Note that the triangle inequality tells us that for $z, z' \in E_{N-2}$ we have

$$\sup_{n \leq N-2} d(T^n z, T^n z') < \delta.$$

Since $T^N = I$ we see that any separation must occur at the $(N-1)^{st}$ step. Hence for each $z \in E_{N-2}$ and almost every $z' \in E_{N-2}$ we must have

$$d(T^{N-1} z, T^{N-1} z') > \delta.$$

But now by applying the same argument as in Case 1 we obtain an uncountable collection a disjoint balls which leads to the same contradiction. Thus we have shown that no sensitive metric exists. \square

Note that the class of rational rotations on S^1 with Lebesgue measure falls into this category. This makes sense as the transformations which we consider to have the least amount of mixing (other than the identity) are the rational rotations. Hence it makes sense that these transformations admit no sensitive metrics. The proof however, which relies on Zorn's lemma, is very complicated given the simplicity of the result. This may be taken as indicating that the class of compatible metrics is large.

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